

Evidence for Duality of Conifold from Fundamental String

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Abstract

We study the spectrum of BPS D5-D3-F1 states in type IIB theory, which are proposed to be dual to D4-D2-D0 states on the resolved conifold in type IIA theory. We evaluate the BPS partition functions for all values of the moduli parameter in the type IIB side, and find them completely agree with the results in the type IIA side which was obtained by using Kontsevich-Soibelman's wall-crossing formula. Our result is a quite strong evidence for string dualities on the conifold.

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1 Introduction and summary

String dualities give us many interesting equivalences between different expressions for string theory. For example, the T-duality relates type IIA and IIB string theory, where the background fields in both sides are non-trivially related to each other. The S-duality is a more non-trivial example of the string dualities, which relates the weak and strong coupling regimes of type IIB string theory. Since it involves the strong coupling regime in one side, in order to test the S-duality we need some non-perturbative analysis beyond the perturbative string theory.

One of the promising way to test the S-duality is to examine BPS states. In general, BPS states are important probes of non-perturbative properties of the theories with extended ($\mathcal{N} \geq 2$) supersymmetry. The BPS states are generically stable under the change of the hyper multiplet moduli parameters, such as the string coupling constant. However, their degeneracy can actually “jump” at some codimension one subspace in the vector multiplet moduli space. This is called wall-crossing phenomena, which make it valuable and interesting to study counting the BPS states. The BPS counting problem and the wall-crossing phenomena are important also in many other research areas, such as black hole microstates [1, 2, 3, 4, 5, 6, 7, 8, 9, 10], Donaldson-Thomas invariants [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25], topological strings and instanton counting [26, 27, 28, 29, 30, 31, 32], M-theory viewpoint [33, 34], exact counting of $\mathcal{N} = 4$ dyons [35, 36, 37, 38, 39], supersymmetric gauge theories [40, 41, 42, 43] and many others [44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54].

In this paper, we test the string dualities by studying the BPS states of the theory. In particular, we compare the spectra of BPS states in both sides of the duality. In one side, we consider the D4-D2-D0 system in type IIA string theory on the resolved conifold. After taking the T and S-duality, this D-brane system is mapped to a D5-D3-F1 system in type IIB theory on $\mathbb{R}^{1,8} \times S^1$. The original conifold geometry is mapped to two NS5-branes after the T-duality transformation [55, 56] and then the S-duality changes them into two D5-branes. Our purpose in this paper is to check the equivalence of the BPS spectra in both sides of the duality, including the effects of the wall-crossing phenomena.

We start from the type IIA side, where our D-brane bound states of interest are composed of one non-compact D4-brane and various numbers of D2 and D0-branes bound to it. We let our D2-branes be wrapped on the compact two-cycle of the conifold so that the D2-branes have finite mass, while the D4-brane is stretched in a non-compact divisor of the conifold. The BPS partition function is defined by

$$\mathcal{Z}_{\text{BPS}}(u, v) := \sum_{Q_0, Q_2 \in \mathbb{Z}} \Omega(\mathcal{D} + Q_2\beta - Q_0dV) u^{Q_0} v^{Q_2}, \quad (1)$$

where $\Omega(\gamma)$ denotes the BPS index of charge γ and $\mathcal{D} \in H^2(X)$ stands for one unit of the non-compact D4-brane charge. The unit D2 and D0-brane charges are denoted by $\beta \in H^4(X)$ and $-dV \in H^6(X)$ respectively. Therefore $Q_2, Q_0 \in \mathbb{Z}$ are D2 and D0-brane charges of the bound states. The four-form β is dual to the compact two-cycle of the conifold. The Boltzmann weights for D2 and D0 branes are denoted by v and u respectively.

This BPS partition function of the D4-D2-D0 states was already evaluated in [51] by using the Kontsevich-Soibelman's wall-crossing formula [17, 21]. As was previously mentioned, the partition function \mathcal{Z}_{BPS} depends on the moduli parameter z , whose imaginary part $\text{Im } z$ is roughly the size of the compact two-cycle of the conifold.⁵ Suppose we fix $\text{Re } z = 1/2$ and move $\text{Im } z$ from $\text{Im } z = +\infty$ to $\text{Im } z = -\infty$. When the moduli cross the walls of marginal stability, the partition function jumps. For $\text{Im } z > 0$, it is given by⁶

$$\mathcal{Z}_{\text{BPS}}(u, v) = \prod_{k=1}^{\infty} \left(\frac{1}{1 - u^k} \right) \prod_{m=0}^{\infty} (1 - u^m v) \prod_{n=1}^{n_0} (1 - u^n v^{-1}). \quad (2)$$

The integer $n_0 > 0$ denotes the number of wall-crossings that occur when we move the moduli from $\text{Im } z = 0$ to its given value. On the other hand, for $\text{Im } z \leq 0$, the partition function can be written as

$$\mathcal{Z}_{\text{BPS}}(u, v) = \prod_{k=1}^{\infty} \left(\frac{1}{1 - u^k} \right) \prod_{m=m_0}^{\infty} (1 - u^m v), \quad (3)$$

where m_0 is the number of walls between the given value of $\text{Im } z (\leq 0)$ and $\text{Im } z = 0$.

Now, our remaining task is to evaluate the BPS partition function in the dual type IIB side, and compare the result with the above expressions in the type IIA side. This is the main subject of this paper.

Let us here summarize our results in this paper. We study the spectrum of the BPS D5-D3-F1 states in type IIB theory, which are dual to the above D4-D2-D0 states on the conifold. One of the most important advantages of the type IIB side is that the BPS spectrum can be analyzed in the perturbative open string theory on the flat spacetime $\mathbb{R}^{1,8} \times S^1$. The D0 and D2-brane charges in the type IIA

⁵To be more precise, we set z so that the central charge of a D2-brane is equal to z . When $\text{Im } z$ is large, $\text{Re } z$ and $\text{Im } z$ are regarded as the B-field and the size of the compact two-cycle of the conifold.

⁶In [51], $\prod_{k=1}^{\infty} (1 - u^k)^{-1}$ in equation (2) is replaced by $\prod_{k=1}^{\infty} (1 - u^k)^{1 - \chi(C_4)}$ where $\chi(C_4)$ is the Euler characteristics of the four-cycle C_4 wrapped by the D4-brane. Since our D4-brane is non-compact, the Euler characteristics $\chi(C_4)$ generally has an ambiguity. However, in this paper, we omit all such ambiguities coming from the non-compactness and simply set $\chi(C_4) = \chi(\mathbb{P}^1) = 2$.

side are identified with the winding number of the fundamental strings and the electric charge on the D5-brane, respectively. We examine the spectrum of the BPS fundamental strings stretched between D3 and D5-branes, and calculate the BPS partition function. The result is exactly the same as (2) and (3), including the wall-crossing phenomena. This result is a quite strong evidence of the duality.

We also show that the moduli parameters in the type IIA side can be identified with the relative positions of the D5-branes and the electric field on the D3-brane in the type IIB side. This gives us the pictorial understanding of the wall-crossing phenomena of the original D4-D2-D0 system.

The construction of this paper is as follows. In Section 2, we discuss the dualizing procedure of our conifold system with D4-D2-D0 branes. We identify the relevant moduli and the charges in the dual picture. Section 3 and section 4 involve the main result of this paper. We count BPS states by investigating BPS open string spectrum on the D5-D3 system and obtain the BPS partition functions in all the chambers. They completely agree with the BPS spectrum in the type IIA side.

2 Sequence of dualities

In this section, we consider the type IIB dual of the D4-D2-D0 states on the conifold, which will turn out to be D5-D3-F1 states. In the original type IIA side, we put one D4-brane on a non-compact holomorphic four-cycle C_4 of the conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$, and consider D2-branes wrapped on the rigid \mathbb{P}^1 and D0-branes localized in the conifold. These D-brane bound states can be seen as charged particles in four-dimensional spacetime transverse to the six-dimensional conifold geometry. Their BPS partition function was evaluated in [51] as (2) and (3).

Below, we show that this brane configuration is dual to a D5-D3-F1 system after the T and S-dual transformations. We also discuss how the moduli parameters of the theory are mapped under the duality transformations.

2.1 Configuration of D-branes and open strings

Our starting point is the D4-D2-D0 bound states in type IIA string theory on the conifold. The brane configuration is summarized in Table 1. Here C_4 denotes a non-compact four-cycle of the resolved conifold, whose topology depends on the moduli parameter z . In fact, C_4 is topologically equivalent to $\mathcal{O}(-1) \rightarrow \mathbb{P}^1$ and \mathbb{C}^2 in the moduli regions $\text{Im } z > 0$ and $\text{Im } z < 0$, respectively. This is related to the flop transition of the conifold, and its relation to the wall-crossing phenomena was discussed in [51].

	x_0	x_1	x_2	x_3	the conifold
D4	○				wrapping on C_4
D2	○				wrapping on \mathbb{P}^1
D0	○				

Table 1: The BPS D-brane configuration of interest in the original type IIA setup. There is one D4-brane wrapped on the non-compact divisor C_4 of the conifold. The D2-branes are wrapped on the rigid \mathbb{P}^1 while the D0-branes are localized in the conifold.

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$x_9(S^1)$
NS5	○	○	○	○	○	○				
NS5'	○	○	○	○			○	○		
D3	○						○	○	○	
D1	○									○

Table 2: The brane configuration after the T-dual transformation. The original conifold geometry is mapped to two NS5-branes, while the D4-brane becomes a D3-brane. The original D2 and D0-branes change into D1-branes.

Now, we take the T-duality transformation along a $U(1)$ orbit of the conifold. Recall that the defining equation of the conifold is

$$xy = zw, \quad (4)$$

which can be interpreted as a \mathbb{C}^* -fibration over \mathbb{C}^2 parameterized by z, w . That is, for given z, w , we can relate x and y via this equation. There is a $U(1)$ orbit in the \mathbb{C}^* fiber, which is generated by

$$x \rightarrow \lambda x, \quad y \rightarrow \lambda^{-1} y, \quad (5)$$

where λ is a complex number with $|\lambda| = 1$. We take the T-duality transformation along this orbit, which leads to two NS5-branes in flat spacetime [55]. The original D2-branes become D1-branes ending on the NS5-branes, while D0-branes change into winding D1-branes in the type IIB side. On the other hand, the D4-brane wrapped on C_4 becomes D3-brane ending on one of the NS5-branes. The brane configuration after the T-dual transformation is summarized in Table 2. Here x^9 -direction is compactified on a circle for the T-duality transformation.

We further perform the S-duality transformation and obtain a D5-D3-F1 system as in Table 3. This brane configuration can be depicted as in Fig. 1. The locations

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$x_9(S^1)$
D5	○	○	○	○	○	○				
D5'	○	○	○	○			○	○		
D3	○						○	○	○	
F1	○									○

Table 3: The brane configuration after the T and S-dual transformations. Now, the whole system only involves D-branes and fundamental strings, which can be analyzed in the perturbative open string theory.

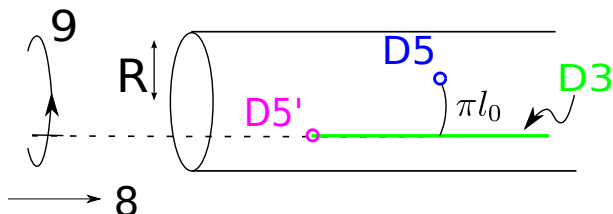


Figure 1: Dualized configuration of D3-D5-D5'-branes. Green line is D-3 brane. Blue and pink points are D5 and D5'-branes respectively. The radius of S^1 in 9 direction is R and the distance between D3 and D5 in 9 direction is πl_0 .

of two D5-branes correspond to two degenerating points of the \mathbb{C}^* -fibration of the conifold, in type IIA language. The D3-brane ends on a D5-brane denoted by D5' because the original D4-brane is wrapped on the divisor $\mathcal{O}(-1) \rightarrow \mathbb{P}^1$ in the conifold.⁷ Our D3-brane is extended to $x^8 = +\infty$ as in figure 1. The fundamental strings are stretched between the D5 and D3-brane. Here, the original D0-brane charge is identified with the winding number of the fundamental strings along x^9 -direction, while the D2-brane charge is mapped to the electric charge on the D5-brane.

Note that this configuration only involves D-branes and fundamental strings in flat spacetime which can be analyzed in the perturbative open string theory. In section 3, we will evaluate the BPS partition function for this brane configuration, by counting the number of stable BPS strings on the D5-D3 system. There are three kinds of relevant open strings, that is, D3-D5, D5-D3 and D3-D3 strings (see Fig. 2).

Let us briefly mention the BPS states without D3-brane. This corresponds to BPS D2-D0 bound states in the type IIA side. There are two types of the BPS

⁷When $\text{Im } z < 0$, the divisor C_4 wrapped by the D4-brane is topologically \mathbb{C}^2 . For the moduli dependence of this brane configuration, see subsection 2.2.

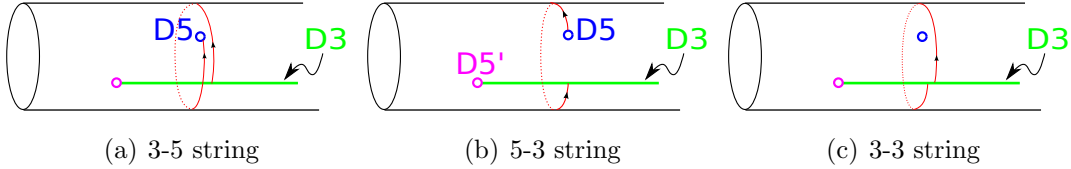


Figure 2: Three kinds of open strings relevant for the BPS counting in the type IIB side. The original D2 and D0-brane charges are now mapped to the electric charge on the D5-brane and the winding number of the strings, respectively.

states: open strings connecting two D5-branes (D5-D5' string) and closed winding strings localized on either D5 or D5'-brane. The D5-D5' string with a winding number becomes a half hyper multiplet in four dimensions and contributes 1 to the BPS index. This number is consistent with the result $\Omega(\pm\beta - ndV) = 1$, $n \in \mathbb{Z}$ in the type IIA side (see for example [15]). On the other hand, the existence of the winding closed strings is consistent with the BPS indices $\Omega(-ndV) = -2$, $n \in \mathbb{Z}$ in the type IIA side.⁸

2.2 Moduli parameters

We here briefly discuss the moduli parameters of the theory. Since our BPS bound states can be seen as charged particles in four dimensions spanned by x^0, x^1, x^2 and x^3 , the spectrum of the BPS states depends on the vector multiplet moduli of the $d = 4, \mathcal{N} = 2$ supersymmetric theory. In the original type IIA setup, this vector multiplet moduli correspond to the Kähler moduli of the conifold, which we denote by z . For large $\text{Im } z$, the real and imaginary parts of z are the B-field and size of the rigid \mathbb{P}^1 of the conifold, respectively.

In order to interpret this moduli z in the type IIB side, let us consider a D2-brane wrapped on the rigid \mathbb{P}^1 of the conifold. Such a D2-brane is mapped to a fundamental string stretched between two D5-branes after the T and S-duality transformations. Since it originally has no D0-brane charge, the dual fundamental string has the vanishing winding number along x^9 -direction. The mass of this fundamental string is, of course, equal to the distance between two D5-branes,⁹ while the mass of the original D2-brane is proportional to $|z|$. So, we find that the distance between two D5-branes in Fig 1 can be written as $|z|$ up to a prefactor. To be more precise, if we set x^8 and x^9 so that the D5' is located at the origin on the x^8 - x^9 plane, the

⁸This counting is actually ambiguous since the internal space is non-compact here.

⁹We can show that there is no non-vanishing B-fields in x^8 - x^9 -plane in the type IIB side, after taking the S-duality transformation.

position of the D5 is determined by

$$x^9 + ix^8 = 2\pi R z, \quad (6)$$

where R is the radius of S^1 on which x^9 -direction is compactified. Note here that the periodicity with respect to the B-field

$$\text{Re } z \sim \text{Re } z + 1 \quad (7)$$

can be understood as a periodicity in the compactified direction x^9 . Thus, the moduli parameter z is interpreted as the position of the D5-brane in x^8 - x^9 plane.¹⁰

One subtlety here is that there is another type of “modulus,” which is a remnant of the Kähler parameters for non-compact cycles of the conifold. In particular, we can turn on the B-field $B \notin \mathbb{Z}$ on the non-compact four-cycle C_4 wrapped by the D4-brane. We choose B to be transverse to the B-field for the rigid \mathbb{P}^1 , namely $\text{Re } z$, and fix it to be non-vanishing as in [51].¹¹ The counterpart of this B-field in the type IIB side can be identified as follows. Since it is transverse to the rigid \mathbb{P}^1 , the T-duality transformation along the \mathbb{C}^* -orbit (5) leaves B invariant. After the T-duality transformation, B can be regarded as a B-field on the D3-brane, whose non-vanishing component is $B_{67} = -B_{76}$. There is a gauge symmetry for the B field and the world volume gauge field A expressed by $B \rightarrow B + d\Lambda$, $A \rightarrow A - \Lambda/(2\pi\alpha')$ with a 1-form parameter Λ . By this gauge transformation, this B-field B_{67} is transformed into a “magnetic” gauge flux F_{67} on the D3-brane.¹² ¹³ Then, the S-duality transformation maps this “magnetic” flux to the “electric” flux F_{08} . Hence, the original B-field for the non-compact cycle C_4 is mapped to the electric flux F_{08} by the T and S-duality transformations.

¹⁰We should here set $x^8 = +2\pi R \text{Im } z$ rather than $x^8 = -2\pi R \text{Im } z$, because we have assumed that the D3-brane is extended to $x^8 = +\infty$ (See Fig. 1). Recall that for $\text{Im } z > 0$ our D4-brane is wrapped on $\mathcal{O}(-1) \rightarrow \mathbb{P}^1$, and therefore two degenerating points of the $U(1)$ -fibration of the conifold are embedded in the divisor wrapped by the D4-brane. This implies that, in the type IIB side, both the D5 and D5'-branes should have the non-vanishing intersection to the D3-brane if $\text{Re } z = 0$ and $\text{Im } z > 0$.

¹¹This new “modulus” can be understood by considering the conifold as a local limit of some compact Calabi-Yau manifold. If we collectively write the Kähler moduli of the (would-be) non-compact cycles as $\Lambda e^{i\varphi}$, only φ remains as a “modulus” after taking the local limit $\Lambda \rightarrow \infty$. This extension of the Kähler moduli of the local Calabi-Yau was proposed in [15]. Note that our non-vanishing B-field for the non-compact cycles implies that $\varphi \notin \pi\mathbb{Z}$, which is the same condition imposed in [51].

¹²This is possible because x_6, x_7 directions are non-compact and thus the magnetic flux is not quantized.

¹³This transformation introduce D1-brane charge (or F-string charge after S-duality) on the D3-brane. However this is irrelevant because we are not counting this D1 or F-string charge.

Before the TS-duality	After the TS-duality
Kähler moduli z for the rigid \mathbb{P}^1	position of D5
B-field B on the non-compact divisor C_4	“electric” field F_{08} on the D3-brane

Table 4: The vector multiplet moduli parameters in both sides of the TS-duality.

In summary, we have one complex moduli parameter z and one real moduli parameter B in the original type IIA setup, which are dual to the position of D5 and the “electric” field on the D3-brane, respectively (see Table 4).

3 Open string spectrum

In this section, we evaluate the spectrum of our BPS bound states with two D5-branes, a single D3-brane, and fundamental strings (Table 3). We first note that the configuration of Table 3 always saturates the BPS bound if there is no fundamental string. This is obvious because such a configuration is dual to a single D4-brane wrapped on a holomorphic cycle in the type IIA side. Putting fundamental strings on the D5-D3 system corresponds to adding D2 and D0-brane charges in the original type IIA setup. Thus, the BPS spectrum of interest can be evaluated by counting the number of BPS string states on the D5-D3 system.

One advantage of our analysis in the dual IIB side is that we only have to analyze the perturbative string theory in *flat* spacetime, while analysis in the type IIA side should take into account non-trivial α' -corrections due to the curved background.¹⁴ It is for this reason that we can explicitly count the number of BPS states for arbitrary values of moduli parameters.

We first note that fundamental strings which we should count is only those on the D3-brane and those stretched between D5 and D3-branes. This is because we are now considering *the bound states* of the D3-brane and some number of fundamental strings; the fundamental strings must be open strings and at least one end of each string must be attached to the D3-brane. The first one will be studied in 3.2, while in 3.1 we analyze the second one.

¹⁴Due to the α' -corrections, it is difficult to evaluate the BPS spectrum directly in the type IIA side. In [51], the authors instead used the Kontsevich-Soibelman formula that tells us how the BPS index changes in wall-crossings.

3.1 Open strings between D5 and D3-branes

By definition, BPS saturated states lie in the bottom of the mass spectrum with fixed charges. Thus, our BPS string states can be identified with the lowest energy states with the winding number along x^9 -direction and electric charge on the D5-brane fixed. Recall here that the zero point energies of string states in NS and Ramond sectors can be written as

$$a_{\text{NS}} = \left(-\frac{1}{48} - \frac{1}{24}\right)(8 - \nu) + \left(\frac{1}{48} + \frac{1}{24}\right)\nu = \frac{\nu}{16} - \frac{8 - \nu}{16}, \quad (8)$$

$$a_{\text{R}} = 0, \quad (9)$$

respectively, where ν denotes the number of dimensions in which one of the open string endpoints has the Dirichlet boundary condition and the other has the Neumann boundary condition.

We start from strings stretched between the D5 and D3-branes (see Fig. 2). For simplicity, we first consider the case of $B = F_{08} = 0$ for the electric field on the D3-brane. Recall first that the position of the D5-brane is given by (6) when the D5'-brane is located at $x^8 = x^9 = 0$. Since our D3-brane ends on the D5'-brane and is extended to $x^8 = +\infty$, we should impose $\text{Im } z > 0$ on the moduli parameter z so that the strings between the D5 and D3-branes can saturate the BPS bound.¹⁵ Such strings generally have the winding number n along the compactified S^1 and the electric charge ± 1 on the D5-brane. These two charges correspond to the D0 and D2-brane charges in the original type IIA setup. From Table 3, we find that such open strings have $\nu = 8$ and therefore the lowest energy states for the strings arise from the R-sector. This means that the BPS strings between the D5 and D3-branes are fermions in target space. In fact, the R-sector has two fermionic zero modes for oscillations in x^0 and x^9 -directions (see Table 3), which form the two-dimensional Clifford algebra

$$\{\psi_0^\mu, \psi_0^\nu\} = \eta^{\mu\nu}, \quad (10)$$

for $\mu, \nu = 0, 9$. Thus, before the GSO-projection, the lowest energy states behave as a two-dimensional Dirac fermion in target space, while the GSO-projection reduces a chiral half of them. The chirality operator is now defined by

$$\Gamma = \Gamma^0 \Gamma^9 = 2\psi_0^0 \psi_0^9, \quad (11)$$

and we choose the GSO-projected lowest energy state $|k^0, n\rangle$ so that $\Gamma |k^0, n\rangle = -|k^0, n\rangle$. Here, the charge k^0 denotes the energy of the BPS string, while n is

¹⁵ If we set $\text{Im } z < 0$ with $B = 0$, a string stretched between the D5 and D3-branes cannot be orthogonal to the D3-brane, and therefore breaks all the supersymmetry.

its winding number along x^9 -direction. Note here that there are no other conserved momentum or winding number in x^1, \dots, x^8 directions. For later use, we here define k^9 in terms of the winding number n by

$$k^9 = \frac{1}{2\alpha'} (\pm l_0 + 2nR). \quad (12)$$

Here, the sign \pm depends on the choice of the string orientation, and πl_0 is the distance between the D5 and D3-branes. Recall also that R is the radius of S^1 on which x^9 -direction is compactified (see Fig. 1).

In order to obtain physical ground states, we have to further impose L_0 and G_0 -conditions. In our notation, L_0 and G_0 are given by¹⁶

$$L_0 = \alpha' k_\mu k^\mu + \dots, \quad G_0 = \sqrt{2\alpha'} k_\mu \psi_0^\mu + \dots, \quad (13)$$

where $\mu = 0, 9$ and the dots involve the contributions from oscillations. Hence, the L_0 -condition implies

$$(k^0)^2 = (k^9)^2, \quad (14)$$

while the G_0 -condition gives a constraint

$$0 = (k_0 \psi_0^0 + k_9 \psi_0^9) |k^0, n\rangle = (-k^0 + k^9) \psi_0^0 |k^0, n\rangle. \quad (15)$$

In the second equality of (15), we used the condition $\Gamma |k^0, n\rangle = -|k^0, n\rangle$ and the anti-commutation relation (10). Thus, the L_0 and G_0 -conditions imply that $k^0 = k^9$. This means that the energy of a BPS string stretched between the D5 and D3-branes is determined by its winding number n . Hence, we can conclude that *there is only one fermionic quantum state for each winding number n of the BPS string stretched between the D5 and D3-branes.*

Note here that equation (12) and the condition $k^9 = k^0 \geq 0$ imply that the winding number should satisfy $n \geq 0$ or $n > 0$ according to the orientation of the string. Changing the string orientation corresponds to the charge conjugation for the original D2-branes in the type IIA side. Therefore the winding number n is positive or zero if the original D2-brane charge $Q_2 = 1$, while it should be positive if $Q_2 = -1$. This fact is important when we evaluate the BPS partition function. Hereafter, we call strings for $Q_2 = 1$ “D3-D5 strings” and strings for $Q_2 = -1$ “D5-D3 strings” (see Fig. 2).

¹⁶We here adopt the so-called old covariant quantization, and therefore L_0 and G_0 does not include contributions from ghosts in (13). We also mention that the previous definition (12) of k^9 is just for simplicity of equation (13).

3.2 Open strings on the D3-brane

We here investigate strings on the D3-brane, namely “D3-D3 strings.” With the vanishing winding number, such strings just describe the quantum fluctuations of the D3-brane. But now, since x^9 -direction is compactified on S^1 , there are BPS strings on the D3-brane with the non-vanishing winding number n along the S^1 . The winding number n corresponds to the D0-brane charge Q_0 in the type IIA side. For the non-vanishing $Q_0 = n$, this system again becomes a D5-D3-F1 system, but now it does not have any D2-brane charge. This is because both ends of such strings are on the D3-brane. Hence, these BPS bound states are dual to the D4-D0 states on the resolved conifold in the type IIA side.

Let us now count the number of such BPS strings. For each value n of the winding number there is a supersymmetric string ground state¹⁷, which follows from (8) and (9). Such a single-string configuration contributes u^n to the BPS partition function. However, there are more string configurations that give u^n to the partition function. For example, let us consider two strings on the D3-brane with the winding numbers $n - 1$ and 1, respectively. Such a two-string configuration also contributes u^n to the BPS partition function. In general, for a given value n , there are $p(n)$ configurations of strings on the D3-brane that contribute u^n to the BPS partition function, where $p(n)$ denotes the partition number of n . Then, we can write a generating function of such BPS bound states as¹⁸

$$\sum_{n=0}^{\infty} p(n) u^n = \prod_{n=1}^{\infty} \left(\frac{1}{1 - u^n} \right). \quad (16)$$

This is precisely equal to the BPS partition function of D0-branes bound to a single D4-brane on \mathbb{C}^2 .

3.3 BPS partition function

Having discussed the number of BPS D5-D3-F1 bound states, we can now calculate the BPS partition function. The result we will obtain here is consistent with [51],

¹⁷Counting the states of this contribution includes subtlety; this D3-brane includes non-compact direction and the number of states depends on the boundary condition. However we here naively assume that the “Euler number” of this non-compact space is 1. This is just the same subtlety appeared in the D4-D0 bound states in IIA side.

¹⁸If there is no D3-brane here, then such multi-string configurations have no binding energy and the constituents are not bound to each other. In that case, we should not regard them as bound states. But we now have a D3-brane, and the strings are bound to the brane. This implies that the BPS multi-string configurations on the D3-brane can be seen as BPS bound states, and we should count all of them to evaluate the generating function.

which supports the validity of the TS-duality of the conifold.

To evaluate the BPS partition function, we use the following facts:

1. BPS strings between the D5 and D3-branes with the winding number n carry the D0-brane charge $Q_0 = n$. The possible values of n depends on their D2-brane charge Q_2 , that is, $n \geq 0$ for $Q_2 = 1$ while $n < 0$ for $Q_2 = -1$. We call strings for $Q_2 = 1$ “D3-D5 strings” and those for $Q_2 = -1$ “D5-D3 strings.”
2. Strings on the D3-brane carry the vanishing D2-brane charge. Their contributions to the partition function has been evaluated as (16).
3. The number of fermionic BPS states with charge γ contributes to the Witten index $\Omega(\gamma)$ with a minus sign.

Combining these facts, we can evaluate the BPS partition function (1) in our type IIB setup. Let us first evaluate the contribution from the D3-D5 strings. Such strings have the winding number $n = 0, 1, 2, \dots$, each of which contributes $-u^n v$ to the partition function. Recall here that BPS D3-D5 strings are spacetime fermions. In general, we can consider multi-string configurations of such fermionic D3-D5 strings, and therefore the total contributions of the D3-D5 strings to the partition function can be written as

$$\prod_{n=0}^{\infty} (1 - u^n v). \quad (17)$$

Similarly, we can evaluate the contributions of the D5-D3 strings. The only one difference is that the winding number n now runs over $n = 1, 2, 3, \dots$. Thus, the D5-D3 strings contribute

$$\prod_{n=1}^{\infty} (1 - u^n v^{-1}) \quad (18)$$

to the partition function. Note that the D5-D3 strings have the different orientation from the D3-D5 strings, and therefore an opposite D2-brane charge $Q_2 = -1$.

Then, taking these together with the contributions (16) from the D3-D3 strings, we find that the full BPS partition function is evaluated as

$$\mathcal{Z}_{\text{BPS}}(u, v) = \prod_{k=1}^{\infty} \left(\frac{1}{1 - u^k} \right) \prod_{m=0}^{\infty} (1 - u^m v) \prod_{n=1}^{\infty} (1 - u^n v^{-1}). \quad (19)$$

This result agrees with the D4-D2-D0 partition function (2) for $n_0 = \infty$. The limit $n_0 = \infty$ means that all the walls of marginal stability are crossed when we move the moduli z from $\text{Im } z = 0$ to $\text{Im } z > 0$. The reason for this is that we now do not take

into account the B-field B on the original D4-brane in the type IIA side. By taking it into account, in the next subsection, we will see more concrete correspondence between the type IIA and IIB sides.

4 Wall-crossing phenomena in type IIB side

We now examine how the wall-crossing phenomena can be understood in the type IIB side. For this purpose, let us turn on the B-field along the non-compact cycles in the original type IIA side. As seen in subsection 2.2, the B-field on the divisor wrapped by the D4-brane can be mapped to the electric field F_{08} on the dual D3-brane. Such an electric field modifies the BPS condition of our D5-D3-F1 system, which leads to different “slope” of the fundamental strings stretched between the D5 and D3-branes. We will see that such a modified IIB picture provides a pictorial interpretation of the wall crossing phenomena of our BPS states.

4.1 Effect of electric field

Let us consider the BPS configuration of strings between the D5 and D3-branes under the non-vanishing electric field F_{08} on the D3-brane. Recall that the world-sheet action of a string coupled to the electric field at its boundary is written as¹⁹

$$S = \frac{1}{4\pi\alpha'} \int d\tau d\sigma (\partial_a X^\mu \partial_a X_\mu) + \int d\tau A_m(X) \frac{dX^m}{d\tau} \Big|_{\sigma=0}, \quad (20)$$

where $a = \tau, \sigma$ and $\mu = 0, \dots, 9$. We set $\sigma = 0$ to be the boundary of the string ending on the D3-brane, and therefore m takes values in $m = 0, 6, 7, 8$, i.e. the directions in which our D3-brane is extended. For a constant background field F_{08} , the gauge potential A_m can be written in a suitable gauge as

$$A_m(X) = \frac{1}{2} F_{mn} X^n. \quad (21)$$

The variation of the world-sheet action with respect to X^m leads to the boundary condition,²⁰

$$(\partial_\sigma X_m - 2\pi\alpha' F_{mn} \partial_\tau X^n) \Big|_{\sigma=0} = 0, \quad (22)$$

which particularly implies

$$\partial_\sigma X^8 \Big|_{\sigma=0} = -2\pi\alpha' F_{08} \partial_\tau X^0 \Big|_{\sigma=0}. \quad (23)$$

¹⁹We here write only the bosonic part for simplicity.

²⁰We impose the Dirichlet boundary condition on X^1, \dots, X^5 and X^9 at $\sigma = 0$. For the other boundary $\sigma = \pi$, we impose the Neumann boundary condition on X^0, \dots, X^5 and the Dirichlet boundary condition on X^6, \dots, X^9 , corresponding to the existence of the D5-brane.

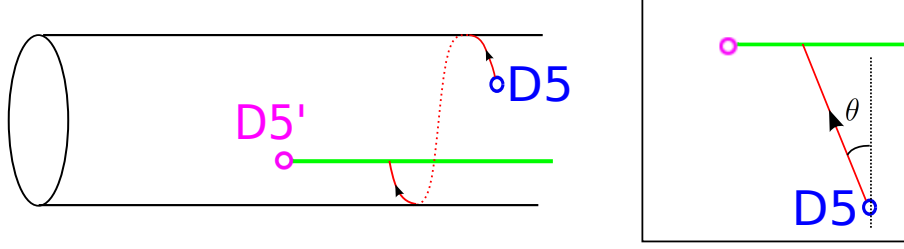


Figure 3: The sketch of a BPS D5-D3 string, which is tilted by the electric field F_{08} in the x^8 - x^9 plane. The green line stands for the D3-brane. The “slope” of the D5-D3 string θ depends on the electric field F_{08} on the D3-brane. If $F_{08} = 0$, then θ vanishes.

From the condition (23), we find that our BPS strings end on the D3-brane with a “slope” depending on the electric field F_{08} . To see this explicitly, we estimate $\delta X^8/\delta X^9$ along the string world-sheet. Note that our D5-D3 strings cannot have non-vanishing momentum in the x^9 -direction. This implies that for BPS D5-D3 strings $X^9(\tau, \sigma)$ has no τ -dependence.²¹ Therefore, we can identify $\delta X^8/\delta X^9$ with $\partial_\sigma X^8/\partial_\sigma X^9$ for BPS D5-D3 strings. In particular, taking the Virasoro constraint $T_{ab} = 0$ into account, this slope can be evaluated at the boundary $\sigma = 0$ as

$$\left. \frac{\delta X^8}{\delta X^9} \right|_{\sigma=0} = \frac{-2\pi\alpha' F_{08}}{\sqrt{1 - (2\pi\alpha' F_{08})^2}} =: -\tan \theta. \quad (24)$$

This quantity vanishes when $F_{08} = 0$, while does not vanish when $F_{08} \neq 0$ ²².

The quantity $\delta X^8/\delta X^9$ represents the “slope” of the BPS strings in x^8 - x^9 plane. If it vanishes, the strings wind along the x^9 -direction and localized in x^8 -direction. This is the case for $F_{08} = 0$, which has already been analyzed in the previous section. On the other hand, if $F_{08} \neq 0$ then the slope $\delta X^8/\delta X^9$ takes a non-zero value, at least at the string boundary on the D3-brane. Since our strings saturate the BPS bound and have no excitations, they should have the constant slope along their world-sheets. This implies that our BPS strings are stretched between the D5 and D3-branes *with a constant slope depending on F_{08}* . The typical example for some non-vanishing F_{08} is depicted in Fig. 3. In the rest of this paper, we set $F_{08} \neq 0$ so that $\delta X^8/\delta X^9 < 0$ is satisfied.²³

²¹For BPS strings, as mentioned before, all the string excitations vanish so that the strings have the lowest energy. For such strings, X^μ depends on τ if and only if it has non-vanishing momentum p^μ .

²²The inequality $|2\pi\alpha' F_{08}| \leq 1$ must be satisfied

²³This corresponds to the condition $\pi/4 < \varphi < \pi/2$ imposed in [51].

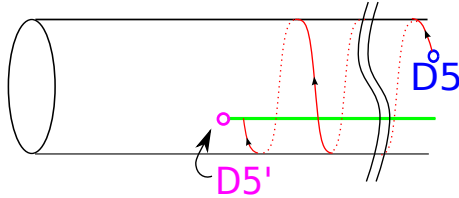


Figure 4: The sketch of the D5-D3 string in case of $\text{Im } z = \infty$, where D3-D5 and D5-D3 strings with arbitrary winding numbers exist in the BPS spectrum.

We here briefly mention the D2-brane charge Q_2 in the type IIA side. As seen at the end of subsection 3.1, the winding number along x^9 -direction is non-negative for both the D5-D3 strings ($Q_2 < 0$) and the D3-D5 strings ($Q_2 > 0$). This and the fact that $\delta X^8 / \delta X^9 < 0$ imply that the string endpoint of $\sigma = \pi$ is always located on the left of the other endpoint $\sigma = 0$, for both the D5-D3 and D3-D5 strings. Therefore, it follows that *D5-D3 strings are extended to the left of the D5-brane while the D3-D5 strings are extended to the right of it*. This fact will be very important when we examine how the results in the previous section are modified under the non-vanishing F_{08} .

4.2 Wall crossing phenomena

We are now ready to see how the wall-crossing phenomena can be understood in the presence of the non-vanishing electric field F_{08} on the D3-brane. We count the number of BPS strings on the D5-D3 system, as in section 3. The only difference from section 3 is the “slope” of the strings induced by the electric field $F_{08} \neq 0$.

Let us first consider the limiting case of $\text{Im } z = +\infty$. Recall that $\text{Im } z$ represents the relative position of the D5-brane in x^9 -direction. So, the limit $\text{Im } z = +\infty$ corresponds to moving the D5-brane far right in Fig. 4. Then, all the BPS D5-D3 and D3-D5 strings considered in the previous section still exist, and the corresponding BPS partition function is the same as (19), that is,

$$\mathcal{Z}_{+\infty}(u, v) = \prod_{k=1}^{\infty} \left(\frac{1}{1 - u^k} \right) \prod_{m=0}^{\infty} (1 - u^m v) \prod_{n=1}^{\infty} (1 - u^n v^{-1}). \quad (25)$$

Next, let us move the moduli parameter $\text{Im } z$ away from $\text{Im } z = +\infty$ while fixing $\text{Re } z$ as $0 < \text{Re } z < 1$.²⁴ For some finite value of $\text{Im } z > 0$, all the D3-D5 strings still remain stable, but some D5-D3 strings disappear from the spectrum. To see this, one considers the BPS D5-D3 string which starts from the D5-brane and extends to

²⁴For example $\text{Re } z = 1/2$ in [51].

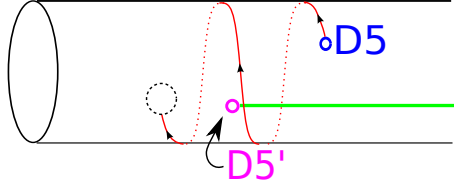


Figure 5: For $\text{Im } z > 0$, some of the D5-D3 strings with large winding numbers does not exist in the BPS spectrum, while all the D3-D5 strings are stable. The maximum winding number n_0 of the D5-D3 strings depends on the moduli $\text{Im } z$.

the left in Fig. 5. Such a D5-D3 string is stretched between the D5 and D3-branes with the fixed slope determined by $F_{08} \neq 0$. This implies that the string endpoint on the D3-brane moves to the left in Fig. 5 as its winding number increases. However, since our D3-brane ends at the D5'-brane, there is a maximum value n_0 of the possible winding number. If a D5-D3 string has the winding number larger than n_0 , the “left” endpoint of the string can not reach the D3-brane while keeping the BPS condition. Therefore, such a D5-D3 string does not exist in the BPS spectrum (see Fig. 5). Thus, we find that the stable BPS D5-D3 string exists if and only if its winding number n is less than or equal to n_0 .

Note here that the maximum winding number n_0 of the D5-D3 strings depends on the moduli $\text{Im } z$. If we keep decreasing $\text{Im } z$, the maximum winding number n_0 decreases by one at $\text{Im } z = (n - \text{Re } z) \tan \theta$, $n \in \mathbb{Z}$ with θ in eq.(24), which means that one D5-D3 string becomes unstable and disappears from the spectrum at each such value of $\text{Im } z$. Such special values of $\text{Im } z$ are on the walls of marginal stability (see Fig. 6). At each such value of $\text{Im } z$, the BPS D5-D3 string with winding number n_0 can marginally decay into a D5-D5' string.²⁵ This is nothing but the wall-crossing phenomenon! Let us call the chamber expressed by

$$(n_0 - \text{Re } z) \tan \theta < \text{Im } z < (n_0 + 1 - \text{Re } z) \tan \theta \quad (26)$$

the “ n_0 -th chamber.”

When $\text{Im } z > 0$, only D5-D3 strings have such wall-crossing phenomena and all the D3-D5 strings are always stable. Therefore, the BPS partition function is written in the n_0 -th chamber ($n_0 > 0$) as

$$\mathcal{Z}_{n_0}(u, v) = \prod_{k=1}^{\infty} \left(\frac{1}{1 - u^k} \right) \prod_{m=0}^{\infty} (1 - u^m v) \prod_{n=1}^{n_0} (1 - u^n v^{-1}). \quad (27)$$

²⁵ This corresponds, in the original type IIA side, to the separation of a D2-D0 fragment from a D4-D2-D0 bound state.

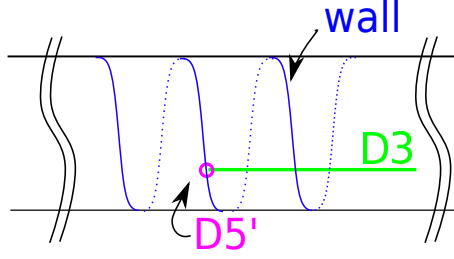


Figure 6: The moduli space and the walls of marginal stability.

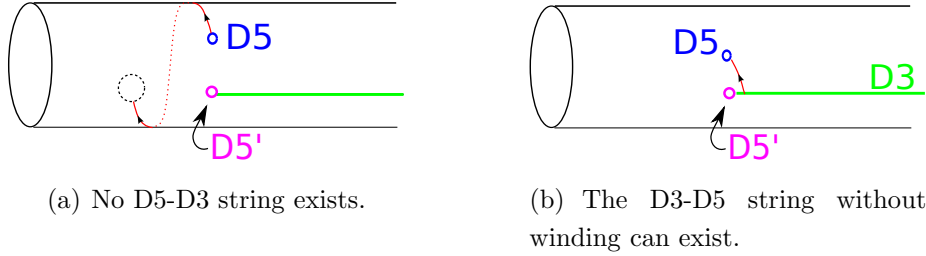


Figure 7: At $\text{Im } z = 0$, we have no D5-D3 strings. So all we have are D3-D5 strings and D3-D3 strings.

In the 0-th chamber, all the D5-D3 strings become unstable (see the left picture of Fig. 7), and the corresponding BPS partition function becomes

$$\mathcal{Z}_0 = \prod_{k=1}^{\infty} \left(\frac{1}{1-u^k} \right) \prod_{m=0}^{\infty} (1-u^m v). \quad (28)$$

These results perfectly agree with equation (2) which was obtained by the analysis in the original type IIA side.²⁶

Let us next consider the case of negative $\text{Im } z$. When we keep decreasing $\text{Im } z$ in the region of $\text{Im } z < 0$, some of the BPS D3-D5 strings might, in turn, be unstable and disappear from the spectrum at some values of $\text{Im } z$. This is because, when $\text{Im } z$ is negative, the D5-brane is located on the left side of the D5'-brane in Fig. 8, which implies that there is a minimum value m_0 of the winding number of the BPS D3-D5 strings. Actually $m_0 = |n_0|$ in the n_0 -th chamber ($n_0 < 0$).²⁷ Thus, the corresponding BPS partition function in the n_0 -th chamber ($n_0 < 0$) is now given by

$$\mathcal{Z}_{n_0}(u, v) = \prod_{k=1}^{\infty} \left(\frac{1}{1-u^k} \right) \prod_{m=|n_0|}^{\infty} (1-u^m v). \quad (29)$$

²⁶ θ in this paper is related to φ in [51] as $\pi - \theta = 2\varphi$.

²⁷Note that all the D5-D3 have already disappeared from the spectrum.

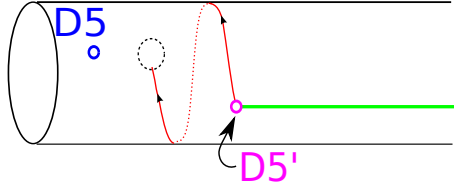


Figure 8: For $\text{Im } z < 0$, there is a minimum value of the winding numbers of D3-D5 strings. The minimum winding number m_0 depends on the moduli $\text{Im } z$.

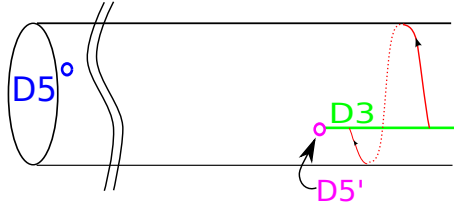


Figure 9: For $\text{Im } z = -\infty$, only the D3-D3 strings exist in the spectrum.

In particular, in the limit of $\text{Im } z = -\infty$, the partition function has no v -dependence:

$$\mathcal{Z}_{-\infty}(u, v) = \prod_{k=1}^{\infty} \left(\frac{1}{1 - u^k} \right). \quad (30)$$

This is because there is no BPS open string stretched between the D5 and D3-branes while only the D3-D3 strings remain stable (see Fig. 9). These results again completely agree with (3) which was obtained from the type IIA analysis.

All the above results perfectly agree with those obtained by the analysis in the original type IIA side, that is, the wall-crossing phenomena of D4-D2-D0 states on the conifold. Namely we obtain the same structures of the walls of marginal stability (see Fig. 10) and the same partition functions in each chamber in both sides. This agreement strongly supports the validity of still mysterious S-duality of string theory.

Acknowledgments

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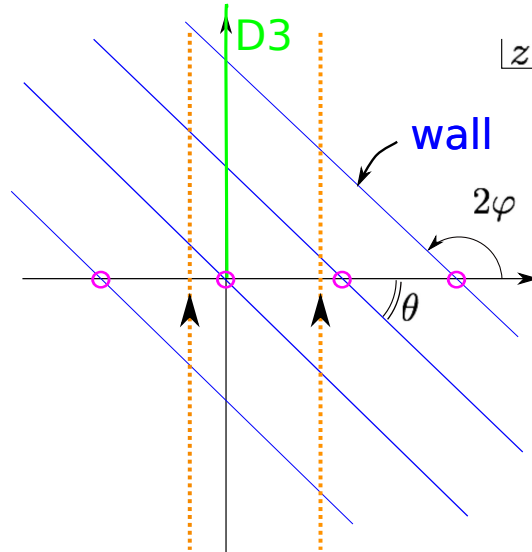


Figure 10: The universal covering of the moduli space and the walls of marginal stability obtained from Fig. 6. Two dotted lines are identified. This picture exactly coincides with [51].

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